PHY 373 Modern Physics II: Quantum Mechanics, Homework Set 4 Solutions

Matthias Ihl

02/28/2007

Note: I will post updated versions of the homework solutions on my homepage: `http://zippy.ph.utexas.edu/~msihl/teaching.html`
We will frequently work in God-given units $c = \hbar = 1$. The casual reader may also want to set $1 = 2 = \pi = -1$.

1 Problem 1

The wave function

$$\langle x|\psi\rangle = \psi(x) = Ne^{ikx}e^{-\frac{x^2}{2\sigma^2}} \tag{1}$$

desribes a wave packet traveling with momentum k in the positive x-direction. (I changed $p \to k$ to avoid confusion! k is arbitrary and has nothing to do with the momentum operator $\hat{p}$ below.)
(a) Normalization:

$$\begin{aligned}\langle\psi|\psi\rangle &= \int_{-\infty}^{+\infty} dx\langle\psi|x\rangle\langle x|\psi\rangle \\ &= \int_{-\infty}^{+\infty} dx\psi^*(x)\psi(x) \\ &= \int_{-\infty}^{+\infty} dxN^2e^{-\frac{x^2}{\sigma^2}} = N^2\sqrt{\pi}\sigma.\end{aligned}$$

Therefore, $N^2 = \frac{1}{\sqrt{\pi}\sigma}$.
(b) Expectation values:

$$\begin{aligned}
\langle \hat{x} \rangle_\psi &= \langle \psi | \hat{x} | \psi \rangle = \int_{-\infty}^{+\infty} dx \langle \psi | \hat{x} | x \rangle \langle x | | \psi \rangle = \int_{-\infty}^{+\infty} dx \; x |\psi(x)|^2 = 0, \\
\langle \hat{p} \rangle_\psi &= \langle \psi | \hat{p} | \psi \rangle = \int_{-\infty}^{+\infty} dx \psi^*(x)(-i\hbar)\frac{\partial}{\partial x}\psi(x) \\
&= \int_{-\infty}^{+\infty} dx \frac{1}{\sqrt{\pi}\sigma}(-i\hbar)(-\frac{x}{\sigma^2} + ik)e^{\frac{-x^2}{\sigma^2}} = \hbar k.
\end{aligned}$$

(c) Uncertainties:

$$\begin{aligned}
\sqrt{\Delta x^2} &= \sqrt{\langle x^2 \rangle} = \sqrt{\int_{-\infty}^{+\infty} dx \; x^2 |\psi(x)|^2} = \frac{\sigma}{\sqrt{2}}. \\
\sqrt{\Delta p^2} &= \sqrt{\langle p^2 \rangle - \hbar^2 k^2} \\
&= \sqrt{\left(\int_{-\infty}^{+\infty} dx \frac{(-i\hbar)^2}{\sqrt{\pi}\sigma}(-\frac{x}{\sigma^2} + ik)^2 e^{\frac{-x^2}{\sigma^2}} \right) - \hbar^2 k^2} \\
&= \sqrt{\hbar^2 k^2 + \left(\int_{-\infty}^{+\infty} dx \; \frac{\hbar^2}{\sqrt{\pi}\sigma^5} x^2 e^{\frac{-x^2}{\sigma^2}} \right) - \hbar^2 k^2} \qquad = \frac{\hbar}{\sqrt{2}\sigma}.
\end{aligned}$$

(d) Probability to find particle on positive x-axis:

$$\int_0^{+\infty} dx \; |\psi(x)|^2 = \frac{1}{2}. \tag{2}$$

2 Problem 2

(a) Normalization: We assume that $\langle 0|0 \rangle = 1$ is properly normalized. Then

$$\begin{aligned}
\langle n|n \rangle &= \langle 0| \frac{a^n}{\sqrt{n!}} \frac{a^{\dagger n}}{\sqrt{n!}} |0\rangle = \frac{1}{n!} \langle 0| a^n a^{\dagger n} |0\rangle \\
&= \frac{1}{n!} \left(\langle 0| a^{n-1} a^{\dagger n} a |0\rangle + \langle 0| a^{n-1} [a, a^{\dagger n}] |0\rangle \right) \\
&= \frac{1}{n!} \langle 0| a^{n-1} [a, a^{\dagger n}] |0\rangle \\
&= \frac{n}{n!} \langle 0| a^{n-1} a^{\dagger n-1} |0\rangle = \frac{1}{(n-1)!} \langle 0| a^{n-1} a^{\dagger n-1} |0\rangle.
\end{aligned}$$

We have used the fact that $[a, a^{\dagger n}] = n a^{\dagger n-1}$ (check!).[1] If we repeat the above step $(n-1)$ times, we arrive at

$$\langle n|n\rangle = \ldots = \frac{2}{2!}\langle 0|aa^\dagger|0\rangle = \langle 0|(a^\dagger a + 1)|0\rangle = \langle 0|0\rangle = 1. \tag{3}$$

Show that $H|n\rangle = \hbar\omega(n + \frac{1}{2})|n\rangle$.

$$\begin{aligned} H|n\rangle &= \hbar\omega(a^\dagger a + \frac{1}{2})\frac{a^{\dagger n}}{\sqrt{n!}}|0\rangle \\ &= \hbar\omega\left(a^\dagger a \frac{a^{\dagger n}}{\sqrt{n!}}\right)|0\rangle + \frac{\hbar\omega}{2}|n\rangle. \end{aligned}$$

Let us look at the first term. Using the commutation relations and defining the number operator $N = a^\dagger a$, we get

$$N a^{\dagger n}|0\rangle = \left(a^{\dagger n} N + [N, a^{\dagger n}]\right)|0\rangle. \tag{4}$$

The first term acting on $|0\rangle$ is zero (why?), the second can be analyzed further, yielding

$$\begin{aligned} [N, a^{\dagger n}] &= a^\dagger a a^{\dagger n} - a^{\dagger n} a^\dagger a = a^\dagger a a^{\dagger n} - a^{\dagger n}(aa^\dagger - 1) = a^\dagger a a^{\dagger n} - a^{\dagger n} a a^\dagger + a^{\dagger n} \\ &= a^\dagger a a^{\dagger n} - a^{\dagger(n-1)}(a^\dagger a)a^\dagger + a^\dagger = a^\dagger a a^{\dagger n} - a^{\dagger(n-1)}(aa^\dagger - 1)a^\dagger + a^\dagger \\ &= a^\dagger a a^{\dagger n} - a^{\dagger(n-1)} a a^{\dagger 2} + 2a^\dagger = \ldots = a^\dagger a a^{\dagger n} - a^\dagger a a^{\dagger n} + n a^{\dagger n} = n a^{\dagger n}, \end{aligned}$$

All together, we have

$$\hbar\omega\left(a^\dagger a \frac{a^{\dagger n}}{\sqrt{n!}}\right)|0\rangle = \hbar\omega\frac{1}{\sqrt{n!}}[N, a^{\dagger n}]|0\rangle = \hbar\omega n \frac{a^{\dagger n}}{\sqrt{n!}}|0\rangle = \hbar\omega n|n\rangle, \tag{5}$$

which proves the assertion.
(b) I will only give references to the literature here. Whoever is interested can get a copy of my handwritten solutions. The derivation of the Hermite polynomials either from the stationary Schrödinger equation or from a recursion relation is dicussed in:
Sakurai, p.93

[1] This means that $[a, .]$ acts as a derivation on functions of $a^\dagger$ which a useful fact to keep in mind.

Griffiths, p.55-59
Bohm, Quantum Mechanics: Foundations and Applications, 3rd edition, ch. I.6, p.28-31; ch. II.7, p.84-86.
(c) Expectation value of x^4 in the groundstate:

$$\begin{aligned}\langle 0|x^4|0\rangle &= \left(\frac{\hbar}{2m\omega}\right)^2 \langle 0|(a+a^\dagger)^4|0\rangle \\ &= \left(\frac{\hbar}{2m\omega}\right)^2 \langle 0|(a^2a^{\dagger 2} + aa^{\dagger 2}a + aa^\dagger aa^\dagger + a^\dagger aa^\dagger a + a^\dagger a^2 a^\dagger + a^{\dagger 2}a^2)|0\rangle \\ &= \left(\frac{\hbar}{2m\omega}\right)^2 \langle 0|\big((N+1)^2 + (N+1) + N(N+1) + (N+1)^2 + N^2 \\ &\quad +N(N+1) + N^2 - N\big)|0\rangle \\ &= 3\left(\frac{\hbar}{2m\omega}\right)^2,\end{aligned}$$

since $N|0\rangle = 0$.
(d) We have

$$|\chi(t)\rangle = \frac{1}{\sqrt{3}}\left(e^{-i\frac{\omega}{2}t}|0\rangle + e^{-i\frac{3\omega}{2}t}|1\rangle + e^{-i\frac{5\omega}{2}t}|2\rangle\right). \tag{6}$$

Therefore, $|\langle 3|\chi(T)\rangle|^2 = 0$. On the other hand,
$|\langle 0|\chi(T)\rangle|^2 = |\langle 0|\frac{1}{\sqrt{3}}e^{-i\frac{\omega}{2}t}|0\rangle|^2 = \frac{1}{3}$.
The average value of x for $|\chi(t>0)\rangle$ is given by

$$\begin{aligned}\langle\chi(t)|x|\chi(t)\rangle &= \frac{1}{3}\sqrt{\frac{\hbar}{2m\omega}}(\langle 0| + \langle 1| + \langle 2|)e^{+\frac{i}{\hbar}Ht}(a+a^\dagger)e^{-\frac{i}{\hbar}Ht}(|0\rangle + |1\rangle + |2\rangle) \\ &= \sqrt{\frac{\hbar}{2m\omega}}\frac{2+2\sqrt{2}}{3}\cos\omega t.\end{aligned}$$

3 Problem 3

(a) Normalization:

$$\int_0^a dx\, |\Psi(x,0)|^2 = 1 \Rightarrow\ A = \frac{1}{\sqrt{2}}, \tag{7}$$

since $\psi_1(x)$ and $\psi_2(x)$ are orthonormal.
(b) With $\omega = \frac{\pi^2\hbar}{2ma^2}$, we have

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\sqrt{\frac{2}{a}}\left(\sin(\frac{\pi}{a}x)e^{-i\omega t} + \sin(\frac{2\pi}{a}x)e^{-i4\omega t}\right) \tag{8}$$

Moreover, following Example 2.1 on page 29,

$$|\Psi(x,t)|^2 = \frac{1}{a}\left(\sin^2(\frac{\pi}{a}x) + \sin^2(\frac{2\pi}{a}x)\right) + \frac{2}{a}\sin(\frac{\pi}{a}x)\sin(\frac{2\pi}{a}x)\cos(3\omega t). \tag{9}$$

(c)Expectation value for x:

$$\langle x\rangle = \int_0^a dx\ x|\Psi(x,t)|^2 = \frac{a}{2} - \frac{16a}{9\pi^2}\cos(3\omega t). \tag{10}$$

(d)Expectation value for p:

$$\langle p\rangle = \int_0^a dx\ \Psi^*(-i\hbar\frac{\partial}{\partial x})\Psi(x,t) = \frac{8\hbar}{3a}\sin(3\omega t). \tag{11}$$

(e)We want to find the energy eigenvalues of $\psi_1(x)$ and $\psi_2(x)$:

$$\langle x|H|\psi_i\rangle = E_i\langle x|\psi_i\rangle. \tag{12}$$

With $H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$, the eigenvalues can be calculated to be

$$E_1 = \frac{\hbar^2}{2m}\frac{\pi^2}{a^2}, \tag{13}$$

$$E_2 = \frac{\hbar^2}{2m}\frac{4\pi^2}{a^2}. \tag{14}$$

The probabilities of finding either E_1 or E_2 are given by

$$|\langle\psi_1|\Psi\rangle|^2 = |\int_0^a dx\ \psi_1^*(x,t)\Psi(x,t)|^2 = \frac{1}{2}, \tag{15}$$

$$|\langle\psi_2|\Psi\rangle|^2 = |\int_0^a dx\ \psi_2^*(x,t)\Psi(x,t)|^2 = \frac{1}{2}. \tag{16}$$

Expectation value of H:

$$\langle H\rangle_\Psi = \left(-\frac{\hbar^2}{2m}\right)\int_0^a dx\ \Psi^*(x,t)\frac{\partial^2}{\partial x^2}\Psi(x,t) = \frac{5}{2}\frac{\hbar^2}{2m}\frac{\pi^2}{a^2} = \frac{1}{2}E_1 + \frac{1}{2}E_2. \tag{17}$$

So the expectation value of H turns out to be the weighted sum of the energy eigenvalues, as expected.

4 Problem 4

$$\begin{aligned}(-i\hbar\frac{\partial}{\partial x})^n\langle x|\psi\rangle &= \int_{-\infty}^{+\infty} dp\ (-i\hbar\frac{\partial}{\partial x})^n\langle x|p\rangle\langle p|\psi\rangle \\ &= \int_{-\infty}^{+\infty} dp\ (-i\hbar\frac{\partial}{\partial x})^n e^{\frac{i}{\hbar}px}\langle p|\psi\rangle \\ &= \int_{-\infty}^{+\infty} dp\ (-i\hbar)^n(\frac{ip}{\hbar})^n\langle x|p\rangle\langle p|\psi\rangle \\ &= \int_{-\infty}^{+\infty} dp\ p^n\langle x|p\rangle\langle p|\psi\rangle \\ &= \int_{-\infty}^{+\infty} dp\ \langle x|\hat{p}^n|p\rangle\langle p|\psi\rangle \\ &= \langle x|\hat{p}^n|\psi\rangle.\end{aligned}$$