Quantum State Reduction: Generalized Bipartitions from Algebras of Observables

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Disclaimer

- I work on quantum gravity, but this talk will not (explicitly) be about quantum gravity.
- I care about this problem mainly because I'm interested in the (approximate) emergence of spacetime from a (more) fundamental Hilbert-space description.
  - (Including, but not limited to, a holographic description.)
- But (I hope) the results will be interesting more generally.

Outline

I. State Reduction in QM, QFT, and QG
II. Math Interlude: Matrix Algebras
III. Sketch of the Algorithm
IV. Toy Examples
V. Beyond Algebras
VI. Applications

I. State Reduction in QM, QFT, and QG

- How should we describe the state of a system we have only limited information about/can only perform a limited set of measurements on?
- (The most general answer involves Bayes' Theorem, priors, etc, but I'll restrict to *physical* systems.)
- In QM/QFT we're used to answering as follows: trace/integrate out the degrees of freedom we don't keep track of to arrive at a reduced density matrix.

The Partial-Trace Map

- Start with a Hilbert space $\mathcal{H}$ and a state $|\psi\rangle \in \mathcal{H}$ or $\rho \in \mathcal{L}(\mathcal{H})$.
- *If* the Hilbert space is bipartite, $\mathcal{H} \cong \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$, there is a natural state-reduction map onto mixed states in $\mathcal{L}(\mathcal{H}_A)$, the partial-trace map $\rho \mapsto \rho_A \equiv \mathrm{tr}_{\bar{A}}\, \rho$
- The reduced state ρ_A indeed preserves information about a limited set of measurements on the original state: the expectation values of O_A in this state are the same as those of $O_A \otimes I_{\bar{A}}$ in the full state.
- However, this is not the most general such map.

Warmup: Classical Physics

- Classical microphysics: choice of phase/configuration space, time evolution law (→implies symmetries + conserved quantities)
  - Gas of particles in a box, mass distribution in galaxy, …
- Microstates = points in configuration space
- Arbitrary macrostates = collections of/distributions over microstates ("coarse grainings")
- *Good* macrostates = possible to measure macroscopically, approximately preserved under time evolution (macrostates evolve to macrostates)
  - States with definite values of thermodynamic/hydrodynamic properties, planets/stars, …

The Decoherence Picture

- Now let's try to map this back to the QM picture.
  - Phase space → Hilbert space
  - Macrostate → reduced density matrix
  - Macrostates evolve to macrostates → reduced density matrix remains nearly diagonal in some basis under the action of time evolution
- The (Zurekian) decoherence program, given a system-environment split and a decomposition of the Hamiltonian $H = H_S \otimes I_E + H_{\text{int}} + I_S \otimes H_E$, tells us which initial states and choices of interaction lead to this branching/evolution without interference.
- So a partial-trace map tracing out the environment describes a classical coarse-graining when decoherence occurs.

Beyond the Partial-trace Map

- However, most coarse-grainings cannot be described in the decoherence picture—just the coarse-grainings which preserve observables on a single factor of a bipartite Hilbert space.
  - *Collective* or averaged observables, in particular, don’t take this form but are very natural laboratory quantities.
  - The Hilbert space may not factorize in a simple way. In particular, we can’t apply the partial-trace map to get a good notion of a state restricted to a spatial region in field theories, or theories with global constraints like gauge or gravitational theories.
- We’d like more general state-reduction maps which we can apply in these cases—and which output *bona fide* reduced states so we can compute entropy and check decoherence.

Bipartitions

- Let’s consider what general state-reduction maps from one (space of operators on a) Hilbert space to another look like.
- If we already have a bipartition/factorization that includes the target Hilbert space, this is just a matter of explicitly specifying which states in the original space are mapped to the various basis states in the target space.

- Factorization: $\mathcal{H}_A \otimes \mathcal{H}_B \quad |a_i, b_k\rangle := |a_i\rangle \otimes |b_k\rangle$
- Bipartition table

a_1, b_1	a_1, b_2	$\cdots$	a_1, b_{d_B}
a_2, b_1	a_2, b_2	$\cdots$	a_1, b_{d_B}
$\vdots$	$\vdots$	$\ddots$	$\vdots$
a_{d_A}, b_1	a_{d_A}, b_2	$\cdots$	a_{d_A}, b_{d_B}

- Bipartition operators for each pair of columns

$$S_{kl} := \sum_{i=1\ldots d_A} |a_i, b_k\rangle \langle a_i, b_l| = I \otimes |b_k\rangle \langle b_l|$$

- State reduction $\rho \longmapsto \sum_{k,l=1\ldots d_B} tr\left(S_{kl}\rho\right) |b_l\rangle \langle b_k|$

$$= \sum_{k,l=1\ldots d_B} tr\left(I \otimes |b_k\rangle \langle b_l| \rho\right) |b_l\rangle \langle b_k| = tr_A\left(\rho\right)$$

- Preserves subspace of operators

span $\{I \otimes |b_k\rangle \langle b_l|\} = \{I \otimes O_B \mid O_B \in \mathcal{B}(\mathcal{H}_B)\}$

$$\begin{array}{|c|c|c|c|}
\hline
a_1, b_1 & a_1, b_2 & \cdots & a_1, b_{d_B} \\
\hline
a_2, b_1 & a_2, b_2 & \cdots & a_1, b_{d_B} \\
\hline
\vdots & \vdots & \ddots & \vdots \\
\hline
a_{d_A}, b_1 & a_{d_A}, b_2 & \cdots & a_{d_A}, b_{d_B} \\
\hline
\multicolumn{1}{c}{\downarrow} & \multicolumn{1}{c}{\downarrow} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\downarrow} \\
\hline
b_1 & b_2 & \cdots & b_{d_B} \\
\hline
\end{array}$$

Two-Qubit Example

	0_B	1_B
0_A	00	01
1_A	10	11

	even_B	odd_B
0_A	00	01
1_A	11	10

- Different arrangements of the table → different factorizations/state-reductions

$$V = |0_A\rangle\,|\mathrm{even}_B\rangle\,\langle 00| + |0_A\rangle\,|\mathrm{odd}_B\rangle\,\langle 01| + |1_A\rangle\,|\mathrm{odd}_B\rangle\,\langle 10| + |1_A\rangle\,|\mathrm{even}_B\rangle\,\langle 11|$$

- Maps Bell state $|00\rangle + |11\rangle$ to the unentangled state $|0_A\rangle\,|\mathrm{even}_B\rangle + |1_A\rangle\,|\mathrm{even}_B\rangle$

Generalized Bipartitions

- We can consider arrangements more general than a single rectangular table:

$$
\begin{array}{|c|c|c|cc}
\hline
e^1_{11} & e^1_{12} & \cdots & & \\
\hline
e^1_{21} & e^1_{22} & \cdots & & \\
\hline
\vdots & \vdots & \ddots & & \\
\hline
\multicolumn{3}{c|}{} & \multicolumn{1}{c|}{e^2_{11}} & \multicolumn{1}{c|}{\cdots} \\
\cline{4-5}
\multicolumn{3}{c|}{} & \multicolumn{1}{c|}{\vdots} & \multicolumn{1}{c|}{\ddots} \\
\cline{4-5}
\multicolumn{5}{c}{} \\
\multicolumn{5}{r}{\ddots}
\end{array}
$$

$$\mathcal{H} \cong \bigoplus_q \mathcal{H}_{A_q} \otimes \mathcal{H}_{B_q}$$

- (We can also consider general non-rectangular tables, but for most of this talk I'll restrict to the case of block-diagonal tables.)

3-Spin Example

$$\mathcal{H} = \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \cong \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}$$

$\frac{3}{2}, \frac{3}{2}$	$\frac{3}{2}, \frac{1}{2}$	$\frac{3}{2}, -\frac{1}{2}$	$\frac{3}{2}, -\frac{3}{2}$				
				$\frac{1}{2}, \frac{1}{2}, 1$	$\frac{1}{2}, -\frac{1}{2}, 1$		
						$\frac{1}{2}, \frac{1}{2}, 2$	$\frac{1}{2}, -\frac{1}{2}, 2$

$$\mathcal{H} \cong \frac{3}{2} \oplus \left(\mathcal{N}_{\frac{1}{2}} \otimes \frac{1}{2} \right)$$

$\frac{3}{2}, \frac{3}{2}$	$\frac{3}{2}, \frac{1}{2}$	$\frac{3}{2}, -\frac{1}{2}$	$\frac{3}{2}, -\frac{3}{2}$		
				$\frac{1}{2}, \frac{1}{2}, 1$	$\frac{1}{2}, -\frac{1}{2}, 1$
				$\frac{1}{2}, \frac{1}{2}, 2$	$\frac{1}{2}, -\frac{1}{2}, 2$

- The state reduction map is now

$$\rho_B = tr_{(A)}(\rho) := \sum_q \sum_{k,l} tr\left(S^q_{kl}\rho\right) |b^q_l\rangle \langle b^q_k|$$

$$= \sum_q tr_{A_q}(\rho_q) \in \mathcal{L}(\mathcal{H}_B) \qquad\qquad \mathcal{H}_B := \bigoplus_q \mathcal{H}_{B_q}$$

- This is *not* the partial-trace map on $\mathcal{H}$!
- However, we can embed $\mathcal{H}$ into a larger space,

$$\mathcal{H}_A \otimes \mathcal{H}_B := \left(\bigoplus_q \mathcal{H}_{A_q}\right) \otimes \left(\bigoplus_q \mathcal{H}_{B_q}\right) = \bigoplus_{q,q'} \mathcal{H}_{A_q} \otimes \mathcal{H}_{B_{q'}}$$

- In this "diagonal embedding" the partial-trace map tr_A *does* map states in the auxiliary space supported on $\mathcal{H}$ to states in the reduced space.

- To understand what sorts of state-reductions these generalized BPTS are describing, we need to talk about matrix algebras and their irreducible representations.
  - Can equivalently talk about vN algebras, but it will be convenient to have the explicit description of operators as matrices, with particular eigenvalues and eigenspaces, in mind. Will only work explicitly with finite-dimensional cases, where both pictures are identical.

II. Matrix Algebras

Definition 2.1. *A* matrix algebra *is a subset* $\mathcal{A} \subseteq \mathcal{L}(\mathcal{H})$ *such that for any* $M_1, M_2 \in \mathcal{A}$ *and* $c \in \mathbb{C}$*:*

(1) $M_1 + M_2 \in \mathcal{A}$
(2) $M_1 M_2 \in \mathcal{A}$
(3) $cM_1 \in \mathcal{A}$
(4) $M_1^\dagger \in \mathcal{A}$

- Any set of matrices $\mathcal{M} := \{M_1, M_2, ...M_n\}$ generates an algebra $\mathcal{A} := \langle M_1, M_2, ...M_n \rangle$ by taking the closure of the set under the operations in the definition. Note that the algebra includes products, so is *not* just the span of the set.

The Wedderburn Decomposition

Theorem 2.2. *(Wedderburn Decomposition) For every algebra $\mathcal{A} \subseteq \mathcal{L}(\mathcal{H})$, the Hilbert space $\mathcal{H}$ decomposes into*

$$\mathcal{H} \cong \left[\bigoplus_q \mathcal{H}_{A_q} \otimes \mathcal{H}_{B_q} \right] \oplus \mathcal{H}_0$$

such that every element $M \in \mathcal{A}$ is of the form

$$M = \left[\bigoplus_q I_{A_q} \otimes M_{B_q} \right] \oplus 0,$$

where I_{A_q} is the identity on $\mathcal{H}_{A_q}$ and M_{B_q} is any matrix on $\mathcal{H}_{B_q}$, and all matrices of this form are elements of $\mathcal{A}$.

- This is the decomposition of $\mathcal{H}$ into irreps of the algebra $\mathcal{A}$.

- That is, there is some basis for $\mathcal{H}$ where all elements of the algebra are block-diagonal:

$$A = \begin{pmatrix} \underbrace{\begin{matrix} A_1 & & \\ & \ddots & \\ & & A_1 \end{matrix}}_{m_1} & & & \\ & \underbrace{\begin{matrix} A_2 & & \\ & \ddots & \\ & & A_2 \end{matrix}}_{m_2} & & \\ & & \ddots & \\ & & & \underbrace{\begin{matrix} A_s & & \\ & \ddots & \\ & & A_s \end{matrix}}_{m_s} \end{pmatrix} = \begin{pmatrix} I_{m_1} \otimes A_1 & & & \\ & I_{m_2} \otimes A_2 & & \\ & & \ddots & \\ & & & I_{m_s} \otimes A_s \end{pmatrix}$$

$$A_q \in \mathcal{M}\left(n_q, \mathbb{C}\right)$$

Algebras from BPTs

- The decomposition can be described by a *block-diagonal* generalized BPT, with each block giving a product basis for a $\mathcal{N}_q \otimes \mathcal{M}_q$

$$\begin{array}{|c|c|c|c|} \hline e^q_{11} & e^q_{12} & e^q_{13} & e^q_{14} \\ \hline e^q_{21} & e^q_{22} & e^q_{23} & e^q_{24} \\ \hline \vdots & \vdots & \vdots & \vdots \\ \hline e^q_{r1} & e^q_{r2} & e^q_{r3} & e^q_{r4} \\ \hline \end{array} \longrightarrow \qquad \mathcal{N}_q \otimes \mathcal{M}_q$$

- The BPOs form a basis spanning $Alg(\mathcal{O})$, with a simple action under products $S^q_{kl} S^{q'}_{l'k'} = \delta_{ll'}\delta_{qq'} S^q_{kk'}$
- Hence the BPOs are “minimal projections”.

Statement of Our Problem

- So, we’ve seen that the irrep decomposition of a Hilbert space w/r/t an algebra of observables generates a state-reduction map onto a smaller Hilbert space which preserves the expectation values of elements in the algebra.
- Given a set of generators of the algebra, we want a way to explicitly construct the state-reduction map. The main technical result of our paper is an algorithm for accomplishing this.
- First, though, let’s briefly think about where the choice of algebra comes from.

Operational vs. Variational

- In the *operational* picture, we're just given a set of allowed measurements.
- In the decoherence approach, we have in mind that each measurement is implemented by some particular interaction Hamiltonian between our apparatus and the system, and a good measuring apparatus is precisely one for which the "pointer states" of the apparatus are both correlated with system states and classically distinguishable.

- When the irrep decomposition of the Hilbert space with respect to the observables contains multiple terms, we think of the Hilbert space as having different *superselection sectors.* Given our operation constraints, superpositions of states in different sectors are unobservable and unpreparable.
- If we can prepare the system we can also typically let it undergo time evolution, so typically we mean that the Hamiltonian does not mix superselection sectors.

- We could instead ask a different question: what are the algebras which lead to interesting decompositions of a given Hilbert space? This is a variational approach, in which we imagine varying over possible choices of observables, or arrangements of generalized BPTs. Usually we want some compatibility between the decompositions and the Hamiltonian, like in decoherence.
- We can ask, for example, what the “most classical” observables are, provided we have a good measure of this. I’ll return to this question later.

III. Sketch of the Algorithm

- The algebra takes as input a (finite) set of observables acting on a (finite-dimensional) Hilbert space and outputs the generalized BPT which describes the irrep structure of the algebra they generate.
- We can use this BPT to write any element of the algebra in block-diagonal form, or for state reduction.
- We'll construct the BPT by constructing the bipartition operators directly.

Algorithm 1 Irrep decomposition of matrix algebra

procedure IrrepDecomposition($\mathcal{M}$)
    $SpecProjs \leftarrow$ GetAllSpectralProjections($\mathcal{M}$)
    $ReflectNet \leftarrow$ ScatterAllProjections($SpecProjs$)
    $ReflectNet \leftarrow$ EstablishMinimality($ReflectNet$)
    $ReflectNet \leftarrow$ EstablishCompleteness($ReflectNet$)
    $BPT \leftarrow$ ConstructIrrepBasis($ReflectNet$)
    return BPT
end procedure

- I won’t be as explicit here as we are in the paper, and I won’t prove the correctness of each step, just sketch how it works.

Projections

- Instead of working directly with observables, it's convenient to work with the projectors onto distinct eigenspaces given by their spectral decompositions (which generate the same algebra):

$$T^i \overset{\text{spec. dec.}}{\longmapsto} \{\Pi_1^i \ldots\}$$

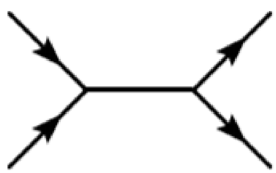
- In general, projectors in the decomposition of one observable will not commute with generators of another observable, so to get a set of BPOs which are all orthogonal with each other we need to decompose these initial projections further.

Scattering

- “Scatter” products of projectors, i.e. decompose them into new projectors.

$$\begin{cases} \Pi^a\Pi^b\Pi^a = & \Pi_1^{ab} + \sum_k \lambda_k \Pi_k^a + 0\Pi_0^a \\ \Pi^b\Pi^a\Pi^b = & \Pi_1^{ab} + \sum_k \lambda_k \Pi_k^b + 0\Pi_0^b \end{cases} \qquad \begin{matrix} \Pi^a \\ \Pi^b \end{matrix} \quad \xmapsto{\text{scatter}} \quad \begin{Bmatrix} \Pi_1^{ab}, \Pi_2^a \ldots \Pi_0^a \\ \Pi_1^{ab}, \Pi_2^b \ldots \Pi_0^b \end{Bmatrix}$$

Π_1 $\quad$ $\Pi_1^{(\lambda_1)} + \Pi_1^{(\lambda_2)} + \ldots + \Pi_1^{(0)}$



Π_2 $\quad$ $\Pi_2^{(\lambda_1)} + \Pi_2^{(\lambda_2)} + \ldots + \Pi_2^{(0)}.$

- The scattering operation reduces rank—the resulting projectors are lower-dimensional and more fine-grained.

Iterating Scattering

- Repeat process until all scattering is trivial (projectors reflecting or orthogonal)

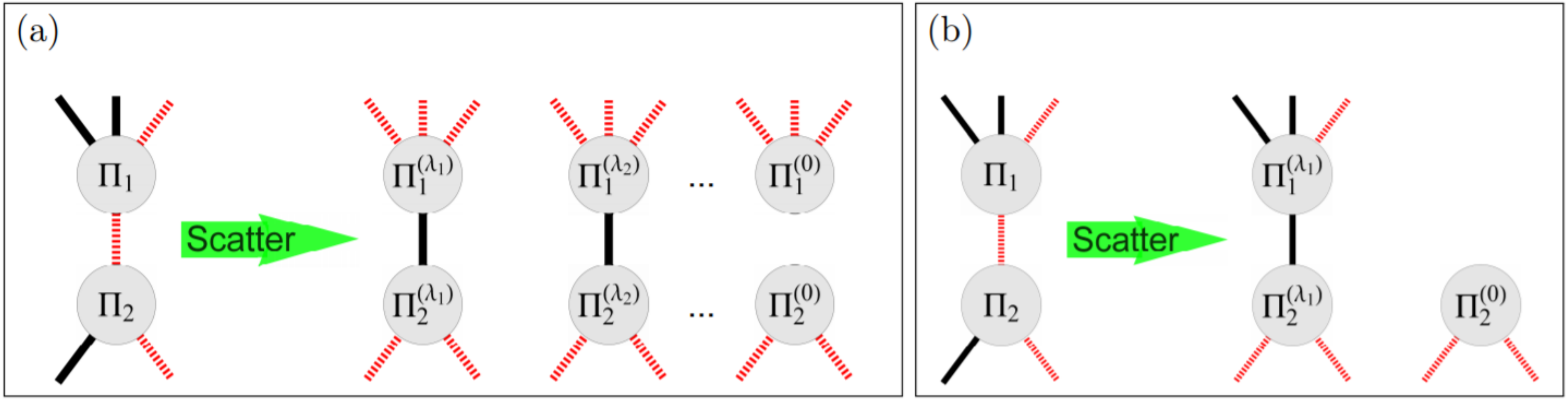
$$\begin{cases} \Pi^a \Pi^b \Pi^a = & \lambda \Pi^a \\ \Pi^b \Pi^a \Pi^b = & \lambda \Pi^b \end{cases}$$

$$\begin{array}{ccc} \Pi^a & \overset{\text{scatter}}{\longmapsto} & \Pi^a \\ \Pi^b & & \Pi^b \end{array}$$

$$\begin{cases} \Pi^a \Pi^b \Pi^a = & 0 \Pi^a \\ \Pi^b \Pi^a \Pi^b = & 0 \Pi^b \end{cases}$$

Graphical Representation

- Define a graph structure ("reflection network"): projectors are connected if they are reflecting, disconnected if orthogonal. Start with the relation between all projectors unknown (dashed line), and update by scattering to resolve each unknown relation:



Minimality and Completeness

- Need additional criteria: all projections in the network should be minimal w/r/t the algebra, and there should exist a subset of the projectors in the network that sums to the identity I_A of the algebra.
- Reduces to checking properties of the network, + adding additional projectors and repeating scattering if necessary—ask me if interested.

Constructing the BPT

- Finally, to construct the BPT, in each connected component we choose a basis for the eigenspace of one projector in the BPT, which forms the first column of the block. Then we construct the remaining columns by traversing the graph between this projector and other projectors in the subset, which defines isometries between the eigenspaces of the projectors.

$$S^q_{kl} = S^q_{k1} S^{q\dagger}_{l1} = S^q_{k1} \Pi_{v^q_1} S^{q\dagger}_{l1} = \sum_{i=1..r_q} S^q_{k1} \left| e^q_{i1} \right\rangle \left\langle e^q_{i1} \right| S^{q\dagger}_{l1} = \sum_{i=1..r_q} \left| e^q_{ik} \right\rangle \left\langle e^q_{il} \right|$$

IV. Toy Examples

- First consider a very simple eight-dimensional model to which we can apply the algorithm.

$$\Pi_{Z;1} := |1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| + |4\rangle\langle 4|$$

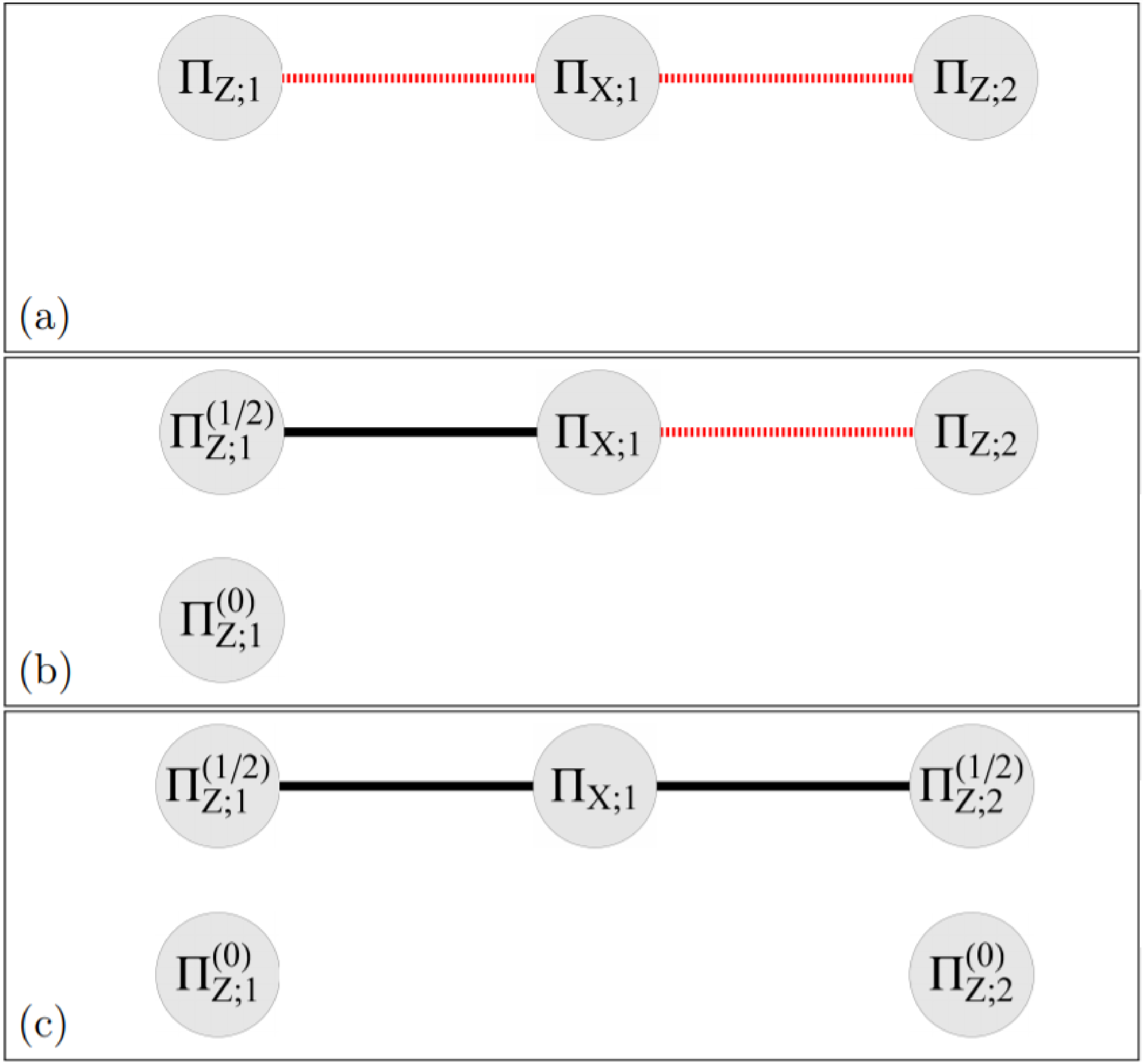
$$\Pi_{X;1} := \left|{}^{+37}_{-}\right\rangle\left\langle{}^{+37}_{-}\right| + \left|{}^{+1256}_{-}\right\rangle\left\langle{}^{+1256}_{-}\right|,$$

$$\left|{}^{+37}_{-}\right\rangle := \frac{1}{\sqrt{2}}\left(|3\rangle + |7\rangle\right)$$

$$\left|{}^{+1256}_{-}\right\rangle := \frac{1}{2}\left(|1\rangle + |2\rangle + |5\rangle + |6\rangle\right)$$

$$\Pi_{Z;2} := I - \Pi_{Z;1}\ ,\ \Pi_{X;2} := I - \Pi_{X;1}$$

$$\langle Z, X\rangle = \langle \Pi_{Z;1}, \Pi_{Z;2}, \Pi_{X;1}, \Pi_{X;2}\rangle$$



$$\Pi_{Z;1}^{(1/2)} = |3\rangle\langle 3| + \left|\substack{+12\\-}\right\rangle\left\langle\substack{+12\\-}\right|$$

$$\Pi_{Z;1}^{(0)} = |4\rangle\langle 4| + \left|\substack{+1\\-2}\right\rangle\left\langle\substack{+1\\-2}\right|$$

$$\Pi_{Z;2}^{(1/2)} = |7\rangle\langle 7| + \left|\substack{+56\\-}\right\rangle\left\langle\substack{+56\\-}\right|$$

$$\Pi_{Z;2}^{(0)} = |8\rangle\langle 8| + \left|\substack{+5\\-6}\right\rangle\left\langle\substack{+5\\-6}\right| .$$

- The reflection network has three connected components. For the single-element components, we’ll choose to use the same basis: the single-column blocks are $\begin{array}{|c|}\hline 4 \\ \hline {}^{+1}_{-2} \\ \hline \end{array}$ and $\begin{array}{|c|}\hline 8 \\ \hline {}^{+5}_{-6} \\ \hline \end{array}$.
- For the three-element component, choose $\left\{\Pi^{(1/2)}_{Z;1}, \Pi^{(1/2)}_{Z;2}\right\}$ as the basis. As before, take $\begin{array}{|c|}\hline 3 \\ \hline {}^{+12}_{-} \\ \hline \end{array}$ as the first column. Then the isometry is

$$S_{21} \propto \Pi^{(1/2)}_{Z;2} \Pi_{X;1} \Pi^{(1/2)}_{Z;1} = \frac{1}{2} \left|7\right\rangle \left\langle 3\right| + \frac{1}{2} \left|{}^{+56}_{-}\right\rangle \left\langle {}^{+12}_{-}\right|$$

so the second column is $\begin{array}{|c|}\hline 7 \\ \hline {}^{+56}_{-} \\ \hline \end{array}$.

- Hence the full BPT is

4			
$+1$ -2			
	8		
	$+5$ -6		
		3	7
		$+12$ $-$	$+56$ $-$

- The Hilbert space decomposition is

$$\mathcal{H} = \mathcal{H}_{A_1} \oplus \mathcal{H}_{A_2} \oplus \mathcal{H}_{A_3} \otimes \mathcal{H}_{B_3}.$$

- All operators in the algebra have the form

$M = c_1 I_{A_1} + c_2 I_{A_2} + I_{A_3} \otimes M_{B_3}$. In particular, write

$Z = a\Pi_{Z;1} + b\Pi_{Z;2}$ and $X = c\Pi_{X;1} + d\Pi_{X;2}$.

- Now we can block-diagonalize the generators by mapping the original basis into the BPT basis:

$$\{|1\rangle,|2\rangle,|3\rangle,|4\rangle,|5\rangle,|6\rangle,|7\rangle,|8\rangle\} \longmapsto \{|4\rangle,\left|{}^{+1}_{-2}\right\rangle,|8\rangle,\left|{}^{+5}_{-6}\right\rangle,|3\rangle,|7\rangle,\left|{}^{+12}_{-}\right\rangle,\left|{}^{+56}_{-}\right\rangle\}$$

$$Z=\begin{pmatrix} a &&&&&&& \\ & a &&&&&& \\ && a &&&&& \\ &&& a &&&& \\ &&&& b &&& \\ &&&&& b && \\ &&&&&& b & \\ &&&&&&& b \end{pmatrix} \longmapsto \begin{pmatrix} a &&&&&&& \\ & a &&&&&& \\ && b &&&&& \\ &&& b &&&& \\ &&&& a &&& \\ &&&&& b && \\ &&&&&& a & \\ &&&&&&& b \end{pmatrix}$$

$$X=\begin{pmatrix} \frac{c+3d}{4} & \frac{c-d}{4} &&& \frac{c-d}{4} & \frac{c-d}{4} && \\ \frac{c-d}{4} & \frac{c+3d}{4} &&& \frac{c-d}{4} & \frac{c-d}{4} && \\ && \frac{c+d}{2} &&&& \frac{c-d}{2} & \\ &&& d &&&& \\ \frac{c-d}{4} & \frac{c-d}{4} &&& \frac{c+3d}{4} & \frac{c-d}{4} && \\ \frac{c-d}{4} & \frac{c-d}{4} &&& \frac{c-d}{4} & \frac{c+3d}{4} && \\ && \frac{c-d}{2} &&&& \frac{c+d}{2} & \\ &&&&&&& d \end{pmatrix} \longmapsto \begin{pmatrix} d &&&&&&& \\ & d &&&&&& \\ && d &&&&& \\ &&& d &&&& \\ &&&& \frac{c+d}{2} & \frac{c-d}{2} && \\ &&&& \frac{c-d}{2} & \frac{c+d}{2} && \\ &&&&&& \frac{c+d}{2} & \frac{c-d}{2} \\ &&&&&& \frac{c-d}{2} & \frac{c+d}{2} \end{pmatrix}$$

Decomposition of Angular Momentum

- Consider a single particle with spin ½ and orbital angular momentum *l*. Of course we know how to decompose the total angular momentum using Clebsch-Gordon coefficients, but we can reproduce this result using scattering of projections.

- Observables $J_r := L_r \otimes I + I \otimes S_r$ for all axes *r*
- Decompose

$$\Pi_{r;m_J} := \begin{cases} \left|r; \pm l, \pm \frac{1}{2}\right\rangle \left\langle r; \pm l, \pm \frac{1}{2}\right| & |m_J| = l + \frac{1}{2} \\ \sum_{m_S = \pm 1/2} \left|r; m_J - m_S, m_S\right\rangle \left\langle r; m_J - m_S, m_S\right| & |m_J| < l + \frac{1}{2}. \end{cases}$$

- Sufficient to consider the algebra generated by $\{J_z, J_x\}$, since rotations $e^{-i\theta J_x}$, etc are in it.
- So we need to scatter projections in the set $\{\Pi_{z;m_J}, \Pi_{x;m_J}\}$. The projectors with maximal/minimal values of *l* are rank 1, so do not break under scattering.

- We compute $\left|Z; l+\frac{1}{2}, m_J\right\rangle := \frac{1}{\sqrt{N_{m_J}}}\Pi_{z;m_J}\left|x; l, \frac{1}{2}\right\rangle$

$$= \left|z; m_J - \frac{1}{2}, \frac{1}{2}\right\rangle \sqrt{\frac{l - m_J - \frac{1}{2}}{2l+1}} + \left|z; m_J + \frac{1}{2}, -\frac{1}{2}\right\rangle \sqrt{\frac{l - m_J + \frac{1}{2}}{2l+1}}$$

$$= \left|z; m_J - \frac{1}{2}, \frac{1}{2}\right\rangle c_-^{l+1, m_J} + \left|z; m_J + \frac{1}{2}, -\frac{1}{2}\right\rangle c_+^{l+1, m_J}.$$

- So the CG result is reproduced, and the BPT is

$l+\frac{1}{2}, l+\frac{1}{2}$	$\cdots$	$l+\frac{1}{2}, -l-\frac{1}{2}$			
			$l-\frac{1}{2}, l-\frac{1}{2}$	$\cdots$	$l-\frac{1}{2}, -l+\frac{1}{2}$

$$\mathcal{H}_L \otimes \mathcal{H}_S \cong \mathcal{H}^{(l+\frac{1}{2})} \oplus \mathcal{H}^{(l-\frac{1}{2})}$$

- Coherences between sectors are not observable.

Collective Observables

- As a toy model of collective observables, we consider a bound pair of identical particles on a lattice of length D, constrained so that their relative position and momentum differ by at most one site. We restrict to center of mass measurements of both position and momentum, and look for the irrep structure of $\langle X_{cm}, P_{cm} \rangle$.

- In the position basis, the momentum states are

$$|p; m_1, m_2\rangle := F\,|x; m_1, m_2\rangle = \frac{1}{D}\sum_{n_1,n_2=0}^{D-1} e^{i2\pi(m_1 n_1 + m_2 n_2)/D}\,|x; n_1, n_2\rangle$$

- The spectral projections are

$$\Pi_{x;n} := |x; n, n+1\rangle\langle x; n, n+1| + |x; n+1, n\rangle\langle x; n+1, n|$$
$$\Pi_{p;m} := |p; m, m+1\rangle\langle p; m, m+1| + |p; m+1, m\rangle\langle p; m+1, m|$$

- So we need to scatter these states.

- A calculation I'll skip shows that $\Pi_{x;n}$ breaks to $\Pi^{(0)}_{x;n}+\Pi^{(\pi)}_{x;n}$, with $\Pi^{(\varphi)}_{x;n} := \left|\chi_n(\varphi)\right\rangle \left\langle \chi_n(\varphi)\right|$, and

similarly for $\Pi^{(\varphi)}_{p;m} := \left|\psi_m(\varphi)\right\rangle \left\langle \psi_m(\varphi)\right|$, with

$$\left|\chi_n(\varphi)\right\rangle := \frac{1}{\sqrt{2}}\left(|x;n,n+1\rangle + e^{i\varphi}|x;n+1,n\rangle\right)$$
$$\left|\psi_m(\varphi)\right\rangle := \frac{1}{\sqrt{2}}\left(|p;m,m+1\rangle + e^{i\varphi}|p;m+1,m\rangle\right)$$

We have

$$\left\langle \chi_n(a\pi) \middle| \psi_m(b\pi) \right\rangle = \sqrt{2}\cos\left(\frac{(b+a)\pi}{2}\right)\left(e^{i2\pi/D} + e^{-ia\pi}\right) e^{i(b+a)\pi/2} e^{i2\pi(2nm+m+n)/D}$$

So for a=0 and b=1, etc, there is no overlap.

- So the reflection network breaks into two components, $\left\{\Pi_{x;n}^{(0)}, \Pi_{p;m}^{(0)}\right\}$ and $\left\{\Pi_{x;n}^{(\pi)}, \Pi_{p;m}^{(\pi)}\right\}$.
- Then the BPT consists of two blocks,

$\chi_0(0)$	$\chi_1(0)$	$\cdots$	$\chi_{D-1}(0)$				
				$\chi_0(\pi)$	$\chi_1(\pi)$	$\cdots$	$\chi_{D-1}(\pi)$

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \cong \mathcal{H}^{(0)} \oplus \mathcal{H}^{(\pi)}$$

- That is, the Hilbert space splits into superselection sectors corresponding to symmetric and antisymmetric configurations: an observer sees a composite particle with a discrete “charge” which is conserved given compatible dynamics.

V. Beyond Algebras

- So far we've worked with block-diagonal BPTs, where the span of the bipartition operators formed an algebra.
- However, in general this need not be the case—operationally, we could imagine we have access to certain observables but not their products.

Partial Bipartitions

- We can consider more general tables, which need not have rectangular blocks:

$e_{1;1,1}$	$e_{1;1,2}$	$\cdots$
$e_{1;2,1}$	$\ddots$	
$\vdots$		

$e_{2;1,1}$	$e_{2;1,2}$	$\cdots$
$e_{2;2,1}$	$\ddots$	
$\vdots$		

$\ddots$

- Now each block still defines a state-reduction map from $\mathcal{H}_q$ to $\mathcal{H}_{B_q}$, which however need not be a tensor factor: we write $\mathcal{H}_q \cong \mathcal{H}_{A_q} \oslash \mathcal{H}_{B_q}$ and say that $\mathcal{H}_{B_q}$ is a *partial subsystem* of $\mathcal{H}_q$.

- The typical example we have in mind is a set of collective degrees of freedom as a partial subsystem, $\mathcal{H} \cong \mathcal{S}_{\text{collective}} \oslash \mathcal{S}_{\text{internal}}$, such as the reduction from a set of N spins to the total spin:

$\frac{N}{2}, +\frac{N}{2}$	$\cdots$	$\frac{N}{2}, +2$	$\frac{N}{2}, +1$	$\frac{N}{2}, 0$	$\frac{N}{2}, -1$	$\frac{N}{2}, -2$	$\cdots$	$\frac{N}{2}, -\frac{N}{2}$
		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
		$2, +2$	$2, +1$	$2, 0$	$2, -1$	$2, -2$		
			$1, +1$	$1, 0$	$1, -1$			
			$\vdots$	$\vdots$	$\vdots$			
			$1, +1$	$1, 0$	$1, -1$			
				$0, 0$				
				$\vdots$				
				$0, 0$				

Ising Model

- There are many possible partial BPTs which reduce onto the same partial subsystem, which we can get by e.g. swapping basis elements within the same column. Such BPTs with different row structure are naturally investigated variationally: different BPTs preserve coherences differently under time evolution, depending on the action of the Hamiltonian.
- See our paper for an Ising model calculation, which I won't have time to talk about today.

V. Applications

- Direct QI applications (error correction, noiseless subsystems, quantum channel design)
  - Won’t discuss here—talk to me if interested!

Bulk Reconstruction

- Think of classical states in AdS/CFT as living in coarse-grained Hilbert space—track only low-point correlation functions of bulk fields
  - → Holographic QEC approach to bulk reconstruction
  - Want to understand how this is implemented in the CFT
- In an explicit toy model (e.g. tensor network), could use our state-reduction methods directly
  - E.g. probe complementary nature of bulk by restricting to observables inside a lightcone
  - Check when "bulk" and "boundary" state-reduction maps yield same output→construct holographic states

Edge Modes

- When a field theory has global constraints (e.g. gauge/global symmetries), physical Hilbert space does not factorize → can't work with usual mode expansion/trace outside subregion
    - Toy example: 3 qubits with Z_2 symmetry
- In edge modes program, define states on subregions by embedding into larger, ungauged Hilbert space (not unique: sum over charged reps)
- Our approach: start with allowed operators, produce state-reduction map (implies diagonal embedding into auxiliary Hilbert space)

Quantum Gravity

- If QG is quantum-mechanical, contains non-field-theoretic states (superpositions of geometries, stringy states, spacetime foam…)
  - …so states well-described around a fixed background are unlikely to be simple factors of the QG Hilbert space (c.f. holography/dS complementarity)
- In "space from Hilbert space" picture, local spatial dofs are emergent
  - GBPs are a tool which precisely picks out dofs not manifest in the full Hilbert space!
  - Dynamics between these dofs + rest of theory can pick out classical observables—variational approach?

…and more

- Potentially many other applications
  - If you have an set of observables you’re interested in, our technology may be able to help! We should chat…

Thank you!

Proposition 2.5. *Let M be a self-adjoint matrix with the spectral decomposition*

$$M = \sum_k \lambda_k \Pi_k$$

where λ_k are distinct (non-zero) eigenvalues and Π_k are projections on eigenspaces. Then

$$\langle M \rangle = \mathsf{span}\left\{\Pi_k\right\}.$$

This fact can be shown by first identifying the identity element $I_{\langle M \rangle}$ in this algebra (it does not have to be the full identity matrix). The identity element is constructed using the minimal polynomial $p(x)$ of M (that is, the smallest degree polynomial for which $p(M) = 0$) and the fact that for self-adjoint matrices the minimal polynomial is of the form $p(x) = f(x)$ or $p(x) = xf(x)$ where f is such that $f(0) \neq 0$. Then

$$I_{\langle M \rangle} := \frac{f(M) - If(0)}{-f(0)} \in \langle M \rangle \tag{2.25}$$

acts as the identity on M, and uniqueness of the identity implies that

$$I_{\langle M \rangle} = \sum_k \Pi_k. \tag{2.26}$$

With the identity, we can re-express the spectral projections as

$$\Pi_k = \prod_{l \neq k} \frac{M - \lambda_l I_{\langle M \rangle}}{\lambda_k - \lambda_l} \in \langle M \rangle. \tag{2.27}$$

Minimality

$$S_{\boldsymbol{v}} := \frac{\Pi_{v_1}\Pi_{v_2}\ldots\Pi_{v_n}}{\sqrt{\lambda_{v_1 v_2}\lambda_{v_2 v_3}\ldots\lambda_{v_{n-1}v_n}}}.$$

Lemma 4.6. *Let $\{\Pi_v\}$ be a set of projections forming a proper reflection network and let $\{S_{\boldsymbol{v}}\}$ be the set of all path isometries in the network as defined by Eq. (4.35). Then, the following statements are equivalent:*

(1) Every Π_v is a minimal projection in the algebra $\mathcal{A} := \langle\{\Pi_v\}\rangle$.
(2) $S_{\boldsymbol{v}} \propto S_{\boldsymbol{u}}$ for all paths $\boldsymbol{v}$, $\boldsymbol{u}$ that share the same initial and final vertices.

Proposition 4.7. *In the setting of Lemma 4.6, let $\boldsymbol{v}$, $\boldsymbol{u}$ be two paths that share the same initial $v_1 = u_1$ and final $v_n = u_m$ vertices but $S_{\boldsymbol{v}} \not\propto S_{\boldsymbol{u}}$. Then, the spectral projections $\{\Pi^{(\omega)}\}$ of $U := S_{\boldsymbol{v}}S_{\boldsymbol{u}}^\dagger$ have the following properties:*

(1) Each $\Pi^{(\omega)}$ is in the algebra $\mathcal{A} := \langle\{\Pi_v\}\rangle$.
(2) Each $\Pi^{(\omega)}$ is not reflecting with Π_{v_1}.

Completeness

Lemma 4.8. *Let $\{\Pi_{v_k}\}$ be the largest subset of pairwise orthogonal projections in the reflection network of $\{\Pi_v\}$, where all Π_v are minimal in the algebra $\mathcal{A} := \langle\{\Pi_v\}\rangle$. If there is a v such that $I_{\mathcal{A}}\Pi_v \neq \Pi_v$, then, with the appropriate normalization factor c, the operator (here I is the full identity matrix and $I_{\mathcal{A}}$ is given by Eq. (4.40))*

$$\tilde{\Pi}_v := \frac{1}{c}\left(I - I_{\mathcal{A}}\right)\Pi_v\left(I - I_{\mathcal{A}}\right), \tag{4.41}$$

has all of the following properties:

(1) $\tilde{\Pi}_v$ is a minimal projection in $\mathcal{A}$.
(2) $\tilde{\Pi}_v$ is orthogonal to all $\{\Pi_{v_k}\}$.
(3) The operator $\tilde{I}_{\mathcal{A}} := I_{\mathcal{A}} + \tilde{\Pi}_v$ is such that $\tilde{I}_{\mathcal{A}}\Pi_v = \Pi_v$.

Partial BPT example

1	2	3
4	5	
	6	

$$\begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} & \rho_{15} & \rho_{16} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} & \rho_{25} & \rho_{26} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} & \rho_{35} & \rho_{36} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} & \rho_{45} & \rho_{46} \\ \rho_{51} & \rho_{52} & \rho_{53} & \rho_{54} & \rho_{55} & \rho_{56} \\ \rho_{61} & \rho_{62} & \rho_{63} & \rho_{64} & \rho_{65} & \rho_{66} \end{pmatrix}$$

$$\downarrow tr_{(A)}$$

$$\begin{pmatrix} \rho_{11} + \rho_{44} & \rho_{12} + \rho_{45} & \rho_{13} \\ \rho_{21} + \rho_{54} & \rho_{22} + \rho_{55} + \rho_{66} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{pmatrix}$$

Ising 1

000	001	**011**	111
	010	**110**	
	100	**101**	

000	010	**011**	111
	001	**101**	
	100	**110**	

000	100	**101**	111
	001	**011**	
	010	**110**	

000	001	**101**	111
	010	**011**	
	100	**110**	

000	010	**110**	111
	001	**011**	
	100	**101**	

000	100	**110**	111
	001	**101**	
	010	**011**	

Table 2: (color online) The 6 selected, quasi-classical BPTs which maximize Q_{BPT} as a measure of dynamical coherence for $N = 3$ spins corresponding to the compatible collective observable $M_c = \sum_{\mu=1}^{3} \sigma_z^{(\mu)}$. Allowed transitions by the Hamiltonian flip single bits in the $\{|0\rangle, |1\rangle\}$ basis. States in the middle two columns not connected by Hamiltonian transitions are shown by the same color.

Ising 2

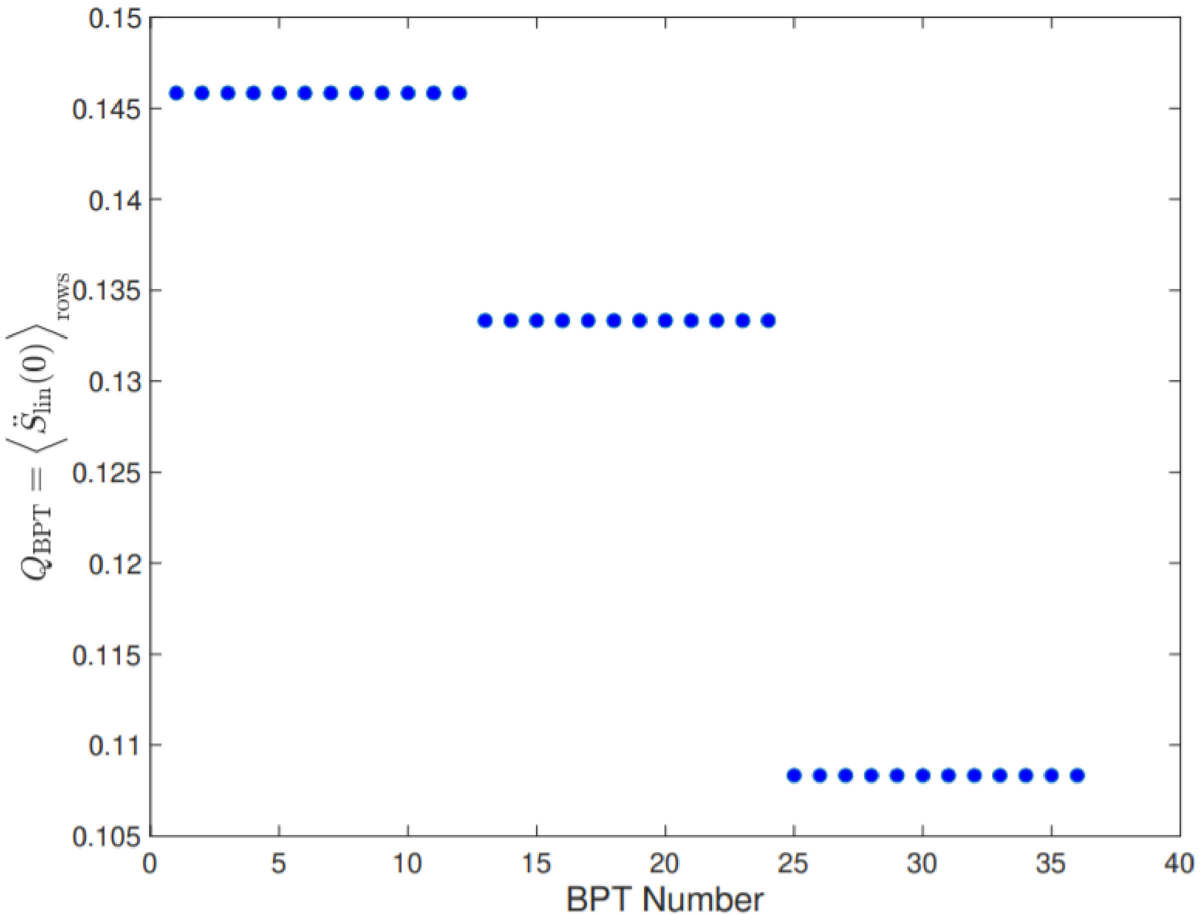


Figure 4: Plot of average entanglement growth rate Q_{BPT} over different BPTs (different row arrangements) for $N = 3$ spins with the compatible collective observable $M_c = \sum_{\mu=1}^{3} \sigma_z^{(\mu)}$ corresponding to a value of $g = 0.5 < g_{\mathrm{crit}}$.

Ising 3

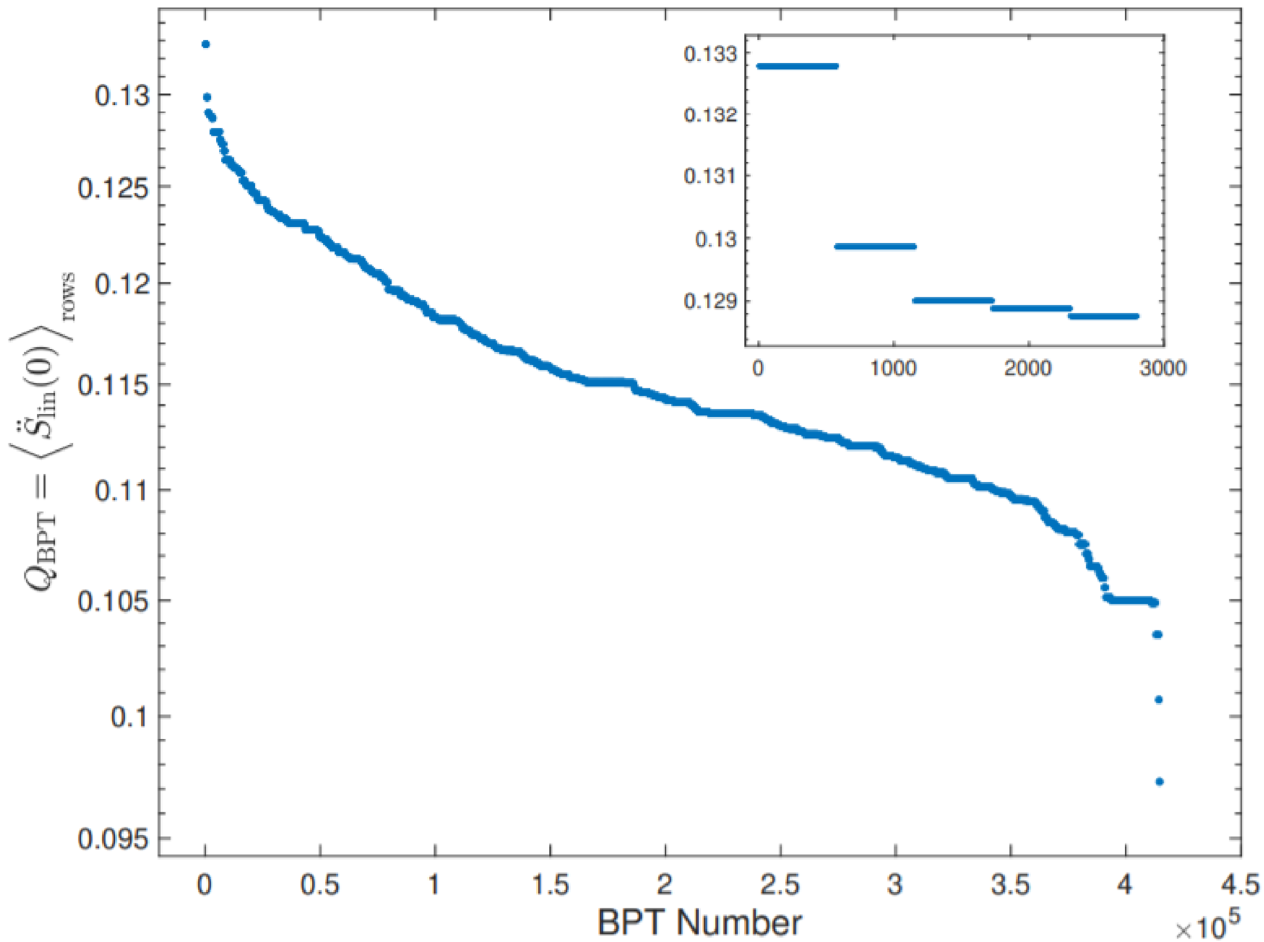


Figure 6: Plot of average entanglement growth rate $Q_{\rm BPT}$ over different BPTs (different row arrangements) for $N = 4$ spins with the compatible collective observable $M_c = \sum_{\mu=1}^{3} \sigma_z^{(\mu)}$ corresponding to a value of $g = 0.6 < g_{\rm crit}$. The inset shows the first few classes of BPTs with lowest values of $Q_{\rm BPT}$.